Journal of Engineering Mathematics **26**: 467–480, 1992.

Tip behaviour for cracks in bonded inhomogeneous materials

P.A. MARTIN

Department of Mathematics, University of Manchester, Manchester M13 9PL, UK

Received 1 November 1991; accepted in revised form 18 February 1992

Abstract. Consider two isotropic elastic half-spaces, welded together with a pressurized crack meeting the weld perpendicularly. A method is developed for determining the asymptotic behaviour of the crack-opening displacement near the tips of the crack. Two problems are considered in detail. In the first, the half-spaces are composed of different homogeneous materials. In the second, the uncracked half-space is inhomogeneous, with a shear modulus that varies exponentially. Each problem is reduced to a hypersingular integral equation, which is then subjected to a Mellin transform. Extensions of the method to other materials and configurations are suggested.

1. Introduction

Consider two isotropic elastic half-spaces, H_c and H_u, bonded together. We choose cartesian coordinates x, y, so that H_c and H_u correspond to the half-planes $x > 0$ and $x < 0$, respectively, and $x = 0$ is the welded interface between H_c and H_u. Let us suppose that there is a crack Γ extending perpendicularly from the interface into H_c; thus, H_c is the *cracked half-space* and H_u is the *uncracked half-space*. Choose the origin so that the crack occupies the line segment

$$\Gamma : 0 \le x \le a, \ y = 0,$$

where a is the length of the crack. Let H_c have shear modulus μ and Poisson's ratio ν, and let H_u have shear modulus μ_u and Poisson's ratio ν_u. We always assume that μ and ν are constants.

We consider a plane-strain deformation caused by a prescribed loading, which is assumed to be symmetric about the x-axis; thus, we suppose that

$$\tau_{yy}(x,0) = \frac{\mu}{1-\nu}\, p(x), \qquad 0 < x < a,$$

where τ_{ij} is the stress tensor and $p(x)$ is (proportional to) the prescribed pressure opening the crack. We are interested in determining the shape of the crack near the tip at $x = 0$. This is related to the stress-intensity factor at the tip.

We consider two problems in detail. In the first (§5), H_u is homogeneous. The crack-opening displacement, $f(x)$, is shown to have an expansion in powers of x; in general, the exponents are neither integers n nor $n + \frac{1}{2}$. The corresponding result for the stress field near the tip at $x = 0$ is well known (see §5 for references); the leading term is proportional to r^α, where $-1 < \alpha < 0$ and r is the distance from the crack tip. This contrasts with the more common inverse square-root behaviour, as for a crack in a homogeneous material. A consequence of this unusual behaviour is that the standard techniques of dynamic fracture mechanics for propagating cracks [1] are not applicable.

One can argue that, in reality, the material parameters (shear modulus and Poisson's ratio) should be continuous across an interface, although they may vary rapidly within an interfacial zone. Recently, Erdogan et al. [2] have modelled this situation by supposing that $\mu_u(x) = \mu e^{\gamma x}$ and $\nu_u = \nu$, where γ is a constant. They found an inverse square-root behaviour near the tip. We study this problem in §6. We confirm the square-root behaviour and go on to find the next term in the expansion of $f(x)$; it involves $\log x$. Such terms do not (usually) arise for homogeneous materials, and are a consequence of the discontinuous derivative of the shear modulus at the interface, $x = 0$.

For each problem, we obtain a hypersingular integral equation for $f(x)$, which we analyse using Mellin transforms, extending the method described in [3]. Thus, we start with minimal assumptions on $f(x)$ and then deduce its asymptotic expansion for small x from the governing integral equation. We remark that Mellin transforms are routinely used for stress-singularity problems in wedges [4, 5]; here, we are interested in more detail (beyond the leading-order contribution) and in the connection between the prescribed loading and the crack-tip behaviour.

2. A crack in an unbounded, homogeneous solid

Consider an unbounded, homogeneous, isotropic, elastic solid, for which we have $\mu_u \equiv \mu$ and $\nu_u \equiv \nu$. The deformation around a pressurized crack Γ in such a solid can be determined by solving a simple hypersingular integral equation, namely

$$\frac{1}{2\pi} ⨍_0^a \frac{f(t)}{(x-t)^2}\,dt = p(x), \qquad 0 < x < a, \tag{2.1}$$

where $f(x)$ is the unknown discontinuity in the normal component of the displacement across the crack at x. The integral must be interpreted as a finite-part integral,

$$⨍_0^a \frac{f(t)}{(x-t)^2}\,dt = \lim_{\varepsilon \to 0} \left\{ \int_0^{x-\varepsilon} \frac{f(t)}{(x-t)^2}\,dt + \int_{x+\varepsilon}^a \frac{f(t)}{(x-t)^2}\,dt - \frac{2f(x)}{\varepsilon} \right\}; \tag{2.2}$$

here, $0 < x < a$ and $f(x)$ is required to have a Hölder-continuous derivative, $f \in C^{1,\alpha}$. The finite-part integral (2.2) is related to a Cauchy principal-value integral by

$$⨍_0^a \frac{f(t)}{(x-t)^2}\,dt = -\frac{d}{dx} ⨍_0^a \frac{f(t)}{x-t}\,dt. \tag{2.3}$$

The integral equation (2.1) was derived by Ioakimidis [6]. It is also seen to be equivalent to *Bueckner's equation* [7, p. 268], if one uses (2.3). Its general solution is [8]

$$f(x) = \frac{2}{\pi} \int_0^a p(t) \log\left(\frac{a|x-t|}{a(x+t) - 2xt + 2\sqrt{xt(a-x)(a-t)}} \right) dt + \frac{A + Bx}{\sqrt{x(a-x)}}, \tag{2.4}$$

where A and B are two arbitrary constants, assuming that p is smooth for $0 < x < a$; p is allowed to have integrable singularities at the two tips. In order to obtain a unique solution, we impose two supplementary conditions on f; these are

$$f(0) = 0 \quad \text{and} \quad f(a) = 0. \tag{2.5}$$

If we impose these edge conditions on f, we must take $A = B = 0$ in (2.4).

For sufficiently well-behaved loadings p, we expect that the asymptotic behaviour of $f(x)$ near the end-points $x = 0$ and $x = a$ is given by

$$f(x) \sim f_1 \sqrt{x} \quad \text{as} \quad x \to 0+, \tag{2.6}$$

and

$$f(x) \sim g_1\sqrt{a-x} \quad \text{as} \quad x \to a-, \tag{2.7}$$

where f_1 and g_1 are constants. In fact, if we restrict p to be square-integrable, rather than merely integrable, we can extract the behaviour of $f(x)$ as $x \to 0$ from (2.4): it is given by (2.6), with

$$f_1 = -\frac{4}{\pi}\int_0^a \frac{\sqrt{a-t}}{\sqrt{at}}\, p(t)\, dt;$$

this is a well known formula [9, p. 90]. A similar formula holds for g_1. Note that the edge behaviour is *not* given by (2.6) if p is *not* square-integrable [10, §29].

It is possible to obtain the complete asymptotic expansion of $f(x)$ as $x \to 0$. One way of doing this is to use the Mellin transform; see §4. This method is described in detail for (2.1) in [3]; the result is

$$f(x) \sim \sum_{n=1} f_n x^{n-1/2} \quad \text{as} \quad x \to 0. \tag{2.8}$$

3. Dissimilar half-spaces

In this section, we describe two problems, corresponding to two different materials in the uncracked half-space H_u. In the first, we suppose that H_u is composed of a homogeneous material, which differs from that in H_c. In the second, we suppose that the material in H_u is inhomogeneous, so that its shear modulus μ_u varies exponentially with x.

3.1. Two homogeneous half-spaces

Suppose that μ_u and ν_u are constants (as well as μ and ν). The associated problem for a pressurized crack Γ has been studied by several authors. It was solved exactly in 1968 by Khrapkov [11] and by Kuang and Mura [12]. They reduced the problem to a singular integral equation, which they solved using Mellin transforms and the Wiener-Hopf technique; the result is rather complicated. Numerical solutions of the singular integral equation have been obtained by Atkinson [13] and by Cook and Erdogan [14]. This integral equation can be written as

$$\frac{1}{2\pi}\int_0^a \left\{\frac{1}{t-x} + H(x,t)\right\} F(t)\, dt = p(x), \qquad 0 < x < a, \tag{3.1}$$

where

$$H(x,t) = \frac{A}{x+t} + \frac{6Bx}{(x+t)^2} - \frac{4Bx^2}{(x+t)^3},$$

$$A = \frac{4(\nu_u - m\nu)(m' - 2m(1-\nu)) - 3m'^2}{\{4(1-\nu_u) - m'\}\{4m(1-\nu) + m'\}},$$

$$B = \frac{m'}{4m(1-\nu) + m'}, \qquad m = \frac{\mu_u}{\mu} \quad \text{and} \quad m' = 1 - m.$$

The integral equation (3.1) is to be solved for F subject to

$$\int_0^a F(t)\, dt = 0.$$

The function F is related to the crack-opening displacement f by

$$f(x) = -\int_x^a F(t)\,dt;$$

thus, $f' = F$ and f satisfies (2.5). An integration by parts converts (3.1) into a hypersingular integral equation for f, namely

$$\frac{1}{2\pi} \oint_0^a \left\{ \frac{1}{(x-t)^2} + L(x,t) \right\} f(t)\,dt = p(x), \qquad 0 < x < a, \tag{3.2}$$

where

$$L(x,t) = -\frac{\partial H}{\partial t} = \frac{A}{(x+t)^2} + \frac{12Bxt}{(x+t)^4}. \tag{3.3}$$

Two special cases are of note. First, if $\mu_u = \mu$, we have $m = 1$, $m' = 0$,

$$A = \frac{\nu - \nu_u}{2(1-\nu_u)} \quad \text{and} \quad B = 0;$$

in particular, if we also have $\nu_u = \nu$, then $A = 0$ and (3.2) reduces to (2.1).
Second, if $\mu_u = 0$, we have $m = 0$, $m' = 1$,

$$A = -1 \quad \text{and} \quad B = 1.$$

In this case, (3.2) reduces to the appropriate integral equation for an edge crack in the stress-free surface of an elastic half-space [15], [3, §§2.4, 3.3].

3.2. *One inhomogeneous half-space*

Suppose that the two half-spaces have the same (constant) Poisson's ratio,

$$\nu_u = \nu.$$

Suppose further that μ_u is given by

$$\mu_u(x) = \mu e^{\gamma x}, \tag{3.4}$$

where γ is a constant; thus, the shear modulus is continuous across the interface, $x = 0$. Erdogan et al. [2] have solved the associated problem for a crack Γ by reducing it to the singular integral equation (3.1), where now

$$H(x,t) = \int_0^\infty h(k,x,t)\, e^{-k(x+t)}\,dk,$$

$$h(k,x,t) = h_1(k) + xh_2(k) + th_3(k) + xth_4(k),$$

and the four functions h_j ($j = 1,2,3,4$) are known algebraic functions of k; see Appendix A. As in §3.1, we integrate by parts to obtain (3.2), where now

$$L(x,t) = \int_0^\infty g(k,x,t)\, e^{-k(x+t)}\,dk, \tag{3.5}$$

$$g(k,x,t) = g_1(k) + xg_2(k) + tg_3(k) + xtg_4(k),$$

and the four functions g_j ($j = 1,2,3,4$) are given by

$$g_1 = kh_1 - h_3, \; g_2 = kh_2 - h_4, \; g_3 = kh_3 \text{ and } g_4 = kh_4.$$

4. Use of Mellin transforms

The Mellin transform is defined by

$$\mathcal{M}f \equiv \widetilde{f}(z) = \int_0^\infty f(x)x^{z-1}\,dx. \tag{4.1}$$

In the sequel, we always use the notation

$$z = \sigma + \mathrm{i}\tau$$

for the transform variable z.

Suppose that $f(x)$ is defined (as the solution of an integral equation) for $0 < x < a$ and satisfies $f(0) = f(a) = 0$. Let us define

$$f(x) = 0 \quad \text{for} \quad x > a,$$

whence $\widetilde{f}(z)$ exists and is analytic in $\sigma \geq 0$; within this right-hand plane,

$$|\widetilde{f}(\sigma + \mathrm{i}\tau)| \to 0 \quad \text{as} \quad |\tau| \to \infty. \tag{4.2}$$

$\widetilde{f}(z)$ can be analytically continued into the left-hand plane $\sigma < 0$, apart from poles. The location and strength of these poles determine the behaviour of $f(x)$ for small x. More precisely, we have the following result [16, p. 7].

THEOREM 1. *Suppose that $\widetilde{f}(z)$ is analytic in a left-hand plane, $\sigma \leq c$, apart from poles at $z = -a_m$, $m = 0, 1, 2, \ldots$, where $\mathrm{Re}(a_0) \leq \mathrm{Re}(a_1) \leq \cdots$; let the principal part of the Laurent expansion of $\widetilde{f}(z)$ about $z = -a_m$ be given by*

$$\sum_{n=0}^{N(m)} A_{mn} \frac{(-1)^n n!}{(z + a_m)^{n+1}},$$

where $N(m)$ is finite. Assume that (4.2) *holds for $c' \leq \sigma \leq c$. Then, if c' can be chosen so that*

$$-\mathrm{Re}(a_{M+1}) < c' < -\mathrm{Re}(a_M)$$

for some M, we have

$$f(x) = \sum_{m=0}^{M} \sum_{n=0}^{N(m)} A_{mn} x^{a_m} (\log x)^n + R_M(x), \tag{4.3}$$

where

$$R_M(x) = \frac{x^{-c'}}{2\pi} \int_{-\infty}^{\infty} \widetilde{f}(c' + \mathrm{i}\tau) x^{-\mathrm{i}\tau}\,d\tau.$$

The remainder $R_M(x)$ is $o(x^{\mathrm{Re}(a_M)})$ if, for example,

$$\int_{-\infty}^{\infty} |\widetilde{f}(c' + \mathrm{i}\tau)|\,d\tau < \infty, \tag{4.4}$$

whence (4.3) *is an asymptotic approximation.*

The inverse Mellin transform is given by

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(z) x^{-z}\, dz.$$

The expansion (4.3) is obtained by moving the inversion contour to the left; each term arises as a residue contribution from an appropriate pole in the analytic continuation of $\tilde{f}(z)$.

There are similar theorems relating the behaviour of a function $p(x)$ for *large* x with the properties of its Mellin transform $\tilde{p}(z)$ in a *right-hand* plane. We shall make use of the following result.

THEOREM 2. *Suppose that*

$$p(x) \sim Ax^{\alpha} \quad \textit{as} \quad x \to 0$$

and

$$p(x) \sim Bx^{-\beta} \quad \textit{as} \quad x \to \infty,$$

where A and B are complex constants, and α and β are real constants with $-\alpha < \beta$. Then, $\tilde{p}(z)$ is analytic for $-\alpha < \sigma < \beta$. Moreover, its analytic continuation into $\sigma \le -\alpha$ has a simple pole at $z = -\alpha$ with residue A, and its analytic continuation into $\sigma \ge \beta$ has a simple pole at $z = \beta$ with residue $-B$.

5. Two homogeneous half-spaces: tip behaviour

Consider the integral equation (3.2), with L defined by (3.3). The function $p(x)$ is given for $0 < x < a$. For simplicity, assume that

$$p(x) = \sum_{n=0} p_n x^n \qquad \text{for small } x; \tag{5.1}$$

other expansions will be considered later. *Define* $p(x)$ for $x > a$ by the left-hand side of (3.2), whence $p(x) = O(x^{-2})$ as $x \to \infty$. Thus, $\tilde{p}(z)$ is analytic for $0 < \sigma < 2$ and can be analytically continued into the whole plane apart from poles. In particular, $\tilde{p}(z)$ has simple poles at $z = -m$ with residue p_m, where $m = 0, 1, 2, \ldots$. Moreover,

$$|\tilde{p}(\sigma + i\tau)| \to 0 \quad \text{as} \quad |\tau| \to \infty \tag{5.2}$$

for all values of σ.

We also know that $f(0) = f(a) = 0$, whence

$$\tilde{f}(z) \quad \text{is analytic for } \sigma \ge 0 \tag{5.3}$$

and has poles in $\sigma < 0$. We locate these poles using the integral equation (3.2).

The left-hand side of (3.2) is a Mellin convolution; in general, we have

$$\mathcal{M}\left\{x^{\alpha} \int_0^{\infty} t^{\beta} k\left(\frac{x}{t}\right) f(t)\, dt\right\} = \tilde{k}(z+\alpha)\tilde{f}(z+\alpha+\beta+1). \tag{5.4}$$

For (3.2), we have $\beta = -2$ and

$$k(x) = \frac{1}{(x-1)^2} + \frac{A}{(x+1)^2} + \frac{12Bx}{(x+1)^4}.$$

In order to calculate $\tilde{k}(z)$, we note that standard contour-integral methods give

$$\fint_0^\infty \frac{x^z}{x-t}\,dx = -\pi t^z \cot \pi z$$

and

$$\int_0^\infty \frac{x^z}{x+t}\,dx = -\frac{\pi t^z}{\sin \pi z}$$

for $-1 < \sigma < 0$ and $t > 0$; differentiation with respect to t (using (2.3)) then gives

$$\mathop{\times\!\!\!\!\!\!\int}_0^\infty \frac{x^z}{(x-t)^2}\,dx = -\pi z t^{z-1} \cot \pi z \quad \text{for} \quad -1 < \sigma < 1,$$

$$\int_0^\infty \frac{x^z}{(x+t)^2}\,dx = \frac{\pi z t^{z-1}}{\sin \pi z} \quad \text{for} \quad -1 < \sigma < 1 \quad \text{and}$$

$$\int_0^\infty \frac{x^{z+1}}{(x+t)^4}\,dx = \frac{\pi z(1-z^2)t^{z-2}}{6 \sin \pi z} \quad \text{for} \quad -2 < \sigma < 2.$$

Hence, with $\alpha = 1$ in (5.4), the Mellin transform of (3.2) is found to be

$$\frac{z}{\sin \pi z} D(z)\tilde{f}(z) = -2\tilde{p}(z+1) \quad \text{for} \quad -1 < \sigma < 1, \tag{5.5}$$

where

$$D(z) = \cos \pi z + 2Bz^2 - (A + 2B). \tag{5.6}$$

We are interested in the zeros of $D(z)$; their properties are described in [11] and [12]. If z is a zero, then so are $-z$, $\bar{z}$ and $-\bar{z}$. $D(z)$ has an infinite number of zeros with $\sigma < 0$. In particular, for any combination of materials, there is always precisely one zero in the strip $-1 < \sigma < 0$, and this zero is real; see Appendix B.

If $B = 0$ (that is, $\mu = \mu_u$), $D(z)$ has an infinite number of real zeros and no complex zeros. However, in general ($B \neq 0$), there is a finite number of real zeros,

$$z = \pm\sigma_n^{\mathrm{r}}, \qquad n = 1, 2, \ldots N,$$

and an infinite number of pairs of complex zeros,

$$z = \pm\sigma_n^{\mathrm{c}} \pm \mathrm{i}\tau_n^{\mathrm{c}}, \qquad n = 1, 2, \ldots;$$

N depends on the values of A and B. Some of these zeros are tabulated in [11] and [12]. For example, suppose $\nu = \frac{1}{4}$ and $\nu_u = \frac{3}{8}$; from [12], we have the following for $m = \frac{1}{3}$

$$N = 2,\ \sigma_1^{\mathrm{r}} = 0.4041,\ \sigma_2^{\mathrm{r}} = 1.2218,\ \sigma_1^{\mathrm{c}} = 2.8072,\ \tau_1^{\mathrm{c}} = 0.8147,$$

whereas for $m = 3$, we have

$$N = 1,\ \sigma_1^{\mathrm{r}} = 0.6353,\ \sigma_1^{\mathrm{c}} = 1.8047,\ \tau_1^{\mathrm{c}} = 0.2863.$$

Returning to (5.5), we have

$$\tilde{f}(z) = -\frac{2\tilde{p}(z+1)\sin \pi z}{zD(z)} \quad \text{for} \quad -1 < \sigma < 1,\ z \neq \pm\sigma_1^{\mathrm{r}}. \tag{5.7}$$

Note that this is not an explicit formula for $\tilde{f}(z)$, since $\tilde{p}$ depends on $\tilde{f}$. Nevertheless, it can be solved, using the Wiener-Hopf technique [11, 12]. We proceed indirectly, and use (5.7) to obtain information on f. In particular, when combined with (5.2), we have

$$\tilde{f}(\sigma + \mathrm{i}\tau) = o(\tau^{-1}) \quad \text{as} \quad |\tau| \to \infty,$$

whence (4.2) and (4.4) hold.

It follows from (5.3) that $\tilde{p}(z+1)$ must have zeros at the simple zeros of $D(z)$ in $\sigma \geq 0$. We then see from (5.7) that $\tilde{f}(z)$ is analytic for $\sigma > -\sigma_1^{\mathrm{r}}$. Take the inversion contour along $\sigma = c$, with $c > -\sigma_1^{\mathrm{r}}$. By Theorem 1, we can move this contour to the left, crossing the first pole at $z = -\sigma_1^{\mathrm{r}}$; this is a simple pole, whence

$$f(x) \sim f_1 x^{\sigma_1^{\mathrm{r}}} \quad \text{as} \quad x \to 0,$$

where

$$f_1 = \frac{2\tilde{p}(1-\sigma_1^{\mathrm{r}}) \sin(\pi\sigma_1^{\mathrm{r}})}{\sigma_1^{\mathrm{r}} D'(\sigma_1^{\mathrm{r}})}.$$

We can continue moving the inversion contour to the left. Noting that the simple poles of $\tilde{p}(z+1)$ are removed by the simple zeros of $\sin \pi z$, we see that $\tilde{f}(z)$ has simple poles at the simple zeros of $D(z)$, whence

$$f(x) \sim \sum_{n=1}^{N} f_n x^{\sigma_n^{\mathrm{r}}} + \sum_{n=1}^{\infty} F_n x^{\sigma_n^{\mathrm{c}}} \cos(\tau_n^{\mathrm{c}} \log x + \delta_n) \quad \text{as} \quad x \to 0, \tag{5.8}$$

where the real quantities F_n and δ_n are defined by

$$F_n e^{i\delta_n} = \frac{-4\tilde{p}(z_n+1) \sin \pi z_n}{z_n D'(z_n)} \quad \text{with } z_n = -\sigma_n^{\mathrm{c}} - \mathrm{i}\tau_n^{\mathrm{c}}.$$

Equation (5.8) gives the complete asymptotic expansion of the crack-opening displacement for loadings of the form (5.1). We note that if $p(x)$ involves non-integer powers, these will induce additional terms in the expansion of $f(x)$. Thus, if $p(x)$ includes a term

$$P x^{\lambda-1},$$

there will be a corresponding term

$$F x^{\lambda}$$

in the expansion of $f(x)$, where the coefficient F is given explicitly by

$$F = \frac{-2P \sin \pi\lambda}{\lambda D(\lambda)}.$$

This will be the leading contribution if $\lambda < \sigma_1^{\mathrm{r}}$, which corresponds to a singular loading, since $\sigma_1^{\mathrm{r}} < 1$.

6. One inhomogeneous half-space: tip behaviour

The governing integral equation is (3.2), namely

$$\frac{1}{2\pi} \int_0^a \frac{f(t)}{(x-t)^2}\, dt + (Lf)(x) = p(x), \qquad 0 < x < a, \tag{6.1}$$

where

$$(Lf)(x) = \frac{1}{2\pi} \int_0^a L(x,t) f(t)\, dt$$

and $L(x,t)$ is defined by (3.5). Suppose that there are constants α and β such that

$$(Lf)(x) = \begin{cases} O(x^{-\alpha-1}) & \text{as } x \to 0, \\ O(x^{-\beta-1}) & \text{as } x \to \infty. \end{cases} \tag{6.2}$$

Then, Theorem 2 implies that $(\widetilde{Lf})(z+1)$ is analytic for $\alpha < \sigma < \beta$; its analytic continuation has simple poles at $z = \alpha$ and at $z = \beta$. Now, proceeding as in §5, we define $p(x)$ for $x > a$ by the left-hand side of (6.1), whence $p(x) = O(x^{-s_2-1})$ as $x \to \infty$, where $s_2 = \min(1,\beta)$. Since $p(x)$ is defined by (5.1) for $0 < x < a$, Theorem 2 implies that $\widetilde{p}(z+1)$ is analytic for $-1 < \sigma < s_2$. Taking the Mellin transform of (6.1) yields

$$z \cot \pi z \, \widetilde{f}(z) - 2(\widetilde{Lf})(z+1) = -2\widetilde{p}(z+1) \quad \text{for} \quad s_1 < \sigma < s_2, \tag{6.3}$$

where $s_1 = \max(-1,\alpha)$.

Let us determine α and β. From (3.5), we have

$$(Lf)(x) = \frac{1}{2\pi} \int_0^\infty \{\mathcal{G}_1(k) + x\mathcal{G}_2(k)\} \, \mathrm{e}^{-kx} \, dk, \tag{6.4}$$

where

$$\mathcal{G}_j = g_j(k)\overline{f}(k) - g_{j+2}(k)\overline{f}'(k)$$

for $j = 1, 2$, and

$$\overline{f}(k) = \int_0^a f(t) \, \mathrm{e}^{-kt} \, dt$$

is the Laplace transform of f (since, by definition, $f(t) = 0$ for $t > a$). From $(2.5)_1$, we know merely that

$$f(x) = o(1) \quad \text{as} \quad x \to 0,$$

but this is sufficient to give [17, p. 134]

$$\overline{f}(k) = O(1) \quad \text{and} \quad \overline{f}'(k) = O(1) \quad \text{as} \quad k \to 0,$$

whereas Watson's lemma [17, p. 103] gives

$$\overline{f}(k) = o(k^{-1}) \quad \text{and} \quad \overline{f}'(k) = o(k^{-2}) \quad \text{as} \quad k \to \infty. \tag{6.5}$$

From the explicit formulae for g_j (see Appendix A), we have

$$g_1(k) = O(k), \; g_2(k) = O(k^2), \; g_3(k) = O(k^2), \; g_4(k) = O(k^3) \quad \text{as} \quad k \to 0,$$

and

$$g_1(k) = O(1), \; g_2(k) = O(k), \; g_3(k) = O(k), \; g_4(k) = O(k^2) \quad \text{as} \quad k \to \infty.$$

Hence

$$\mathcal{G}_1(k) = O(k) \quad \text{and} \quad \mathcal{G}_2(k) = O(k^2) \quad \text{as} \quad k \to 0 \tag{6.6}$$

and

$$\mathcal{G}_1(k) = o(k^{-1}) \quad \text{and} \quad \mathcal{G}_2(k) = o(1) \quad \text{as} \quad k \to \infty. \tag{6.7}$$

Then, since Lf is itself a Laplace transform, we deduce that it behaves as (6.2), wherein

$$\alpha = s_1 = -1 \quad \text{and} \quad \beta = s_2 = 1.$$

Thus, (6.3) gives

$$\tilde{f}(z) = 2\frac{\sin \pi z}{z} \frac{(\widetilde{Lf})(z+1) - \tilde{p}(z+1)}{\cos \pi z} \quad \text{for} \quad -1 < \sigma < 1,\ z \neq \pm\tfrac{1}{2}, \tag{6.8}$$

since $\cos \pi z$ has zeros. We then see from (5.3) that $\tilde{f}(z)$ is analytic for $\sigma > -\frac{1}{2}$, so we take the inversion contour along $\sigma = c$, with $c > -\frac{1}{2}$. Moving this contour to the left, we cross the first pole at $z = -\frac{1}{2}$; this is a simple pole, whence the edge behaviour of f is given by

$$f(x) \sim f_1 \sqrt{x} \quad \text{as} \quad x \to 0, \tag{6.9}$$

with

$$f_1 = (4/\pi)\{(\widetilde{Lf})(\tfrac{1}{2}) - \tilde{p}(\tfrac{1}{2})\}.$$

Thus, the leading edge-behaviour of the crack-opening displacement $f(x)$ is the same as for a crack in an unbounded homogeneous material. This is the result obtained by Erdogan et al. [2]; this paper also contains numerical results for the stress-intensity factor (proportional to f_1) as a function of γa, for $\nu = 0.3$ and $p(x) = p_0$, a constant.

6.1. The next term

Let us now improve the approximation (6.9). First, using (6.9), we can refine (6.5) to

$$\bar{f}(k) \sim \tfrac{1}{2}\sqrt{\pi} f_1 k^{-3/2} \quad \text{and} \quad \bar{f}'(k) \sim -\tfrac{3}{4}\sqrt{\pi} f_1 k^{-5/2} \quad \text{as} \quad k \to \infty.$$

Then, using the known asymptotics of $g_j(k)$ for large k (given in Appendix A), we can refine (6.7) to

$$\mathcal{G}_1(k) \sim \mathcal{A}_1 k^{-3/2} \quad \text{and} \quad \mathcal{G}_2(k) \sim \mathcal{A}_2 k^{-1/2} \quad \text{as} \quad k \to \infty, \tag{6.10}$$

where

$$\mathcal{A}_1 = (1-2\nu)\mathcal{A}_2 \quad \text{and} \quad \mathcal{A}_2 = \frac{\sqrt{\pi}\gamma f_1}{16(1-\nu)}.$$

Next, since $\mathcal{M}\{e^{-kx}\} = k^{-z}\Gamma(z)$, we obtain

$$2\pi(\widetilde{Lf})(z+1) = \Gamma(z+1)\tilde{\mathcal{G}}_1(-z) + \Gamma(z+2)\tilde{\mathcal{G}}_2(-1-z) \tag{6.11}$$

from (6.4). Both $\tilde{\mathcal{G}}_1(-z)$ and $\tilde{\mathcal{G}}_2(-1-z)$ are analytic for $-\frac{3}{2} < \sigma < 1$; this follows from (6.6), (6.10) and Theorem 2. Hence, (6.8) gives

$$\tilde{f}(z) = \frac{1}{\pi z}\left\{\Gamma(z+1)\tilde{\mathcal{G}}_1(-z) + \Gamma(z+2)\tilde{\mathcal{G}}_2(-1-z) - 2\pi\tilde{p}(z+1)\right\} \tan \pi z \tag{6.12}$$

for $\sigma > -\frac{3}{2}$, $z \neq -\frac{1}{2}$; note that the simple poles of $\Gamma(z+1)$ and $\tilde{p}(z+1)$ at the negative integers are removed by corresponding zeros of $\tan \pi z$.

The right-hand side of (6.12) has a pole at $z = -\frac{3}{2}$. Near this pole,

$$\tilde{\mathcal{G}}_1(-z) \sim \frac{\mathcal{A}_1}{z+\frac{3}{2}},\ \tilde{\mathcal{G}}_2(-1-z) \sim \frac{\mathcal{A}_2}{z+\frac{3}{2}} \quad \text{and} \quad \tan \pi z \sim \frac{-1}{\pi(z+\frac{3}{2})},$$

whence $z = -\frac{3}{2}$ is a *double* pole. From (6.12), we have

$$\tilde{f}(z) \sim \frac{\mathcal{B}}{(z+\frac{3}{2})^2} \qquad \text{near } z = -\tfrac{3}{2},$$

where

$$\mathcal{B} = \frac{2}{3\pi^2}\left\{\mathcal{A}_1\Gamma(-\tfrac{1}{2}) + \mathcal{A}_2\Gamma(\tfrac{1}{2})\right\} = \frac{(4\nu - 1)\gamma f_1}{24\pi(1-\nu)}.$$

Theorem 1 then gives

$$f(x) \sim f_1 x^{1/2}\left\{1 + \frac{1-4\nu}{24\pi(1-\nu)}\,\gamma x \log x\right\} + f_2 x^{3/2} \quad \text{as} \quad x \to 0. \tag{6.13}$$

It is not surprising that the logarithmic term vanishes when $\gamma = 0$, for then the two elastic half-spaces become identical and homogeneous; see (2.8). Curiously, the logarithmic term also vanishes when $\nu = \frac{1}{4}$.

7. Discussion

We have examined the behaviour of the crack-opening displacement $f(x)$ for a crack meeting the interface between two isotropic elastic half-spaces. More precisely, we have described a method for obtaining the *exponents* in the asymptotic expansion of $f(x)$ near a crack tip at $x = 0$. In general, the *coefficients* in the expansion can only be obtained by solving the governing integral equation; this can be done more efficiently by incorporating the known crack-tip behaviour of $f(x)$ into an appropriate numerical scheme.

In §5, we considered two different homogeneous half-spaces, and obtained a complete expansion of $f(x)$, namely (5.8). The character of this expansion is unchanged if the two half-spaces have the same Poisson's ratio, $\nu = \nu_u$. However, if $\mu = \mu_u$ (with $\nu \neq \nu_u$), then $N = \infty$ and (5.8) reduces to

$$f(x) \sim \sum_{n=1} f_n x^{\sigma_n^r} \quad \text{as} \quad x \to 0,$$

which should be compared with the known expansion for a crack in a single unbounded homogeneous solid, (2.8). This latter expansion is also obtained if the crack does not meet the interface. To see this, suppose that we locate the interface between the two homogeneous half-spaces at $x = -\varepsilon$, with $\varepsilon > 0$, and take Γ as before. The corresponding integral equation is (3.2), where now

$$L(x,t) = \frac{A}{(x+t+2\varepsilon)^2} + \frac{12B(x+\varepsilon)(t+\varepsilon)}{(x+t+2\varepsilon)^4}.$$

Proceeding as before, we take the Mellin transform and obtain (6.8), where

$$(\widetilde{Lf})(z+1) = \frac{z}{2\sin \pi z}\int_0^a f(t)(t+2\varepsilon)^{z-3}\mathcal{L}(t;z,\varepsilon)\,dt \tag{7.1}$$

and

$$\mathcal{L}(t;z,\varepsilon) = A(t+2\varepsilon)^2 + 2B(t+\varepsilon)(1-z)\{(1+z)(t+2\varepsilon) + \varepsilon(2-z)\}.$$

Provided $\varepsilon > 0$, the integral in (7.1) defines an analytic function of z. It follows that the only contributory poles come from the zeros of $\cos \pi z$ in (6.8), and so we obtain (2.8), just as for a single unbounded homogeneous solid (although, of course, the coefficients f_n will be different). We note that Atkinson [18] has studied the behaviour of the stress field near $x = 0$ as $\varepsilon \to 0$.

In §6, we considered a cracked homogeneous half-space bonded to an inhomogeneous half-space. We confirmed the result of Erdogan et al. [2] that $f(x) = O(\sqrt{x})$ as $x \to 0$. We also obtained the next term, which is proportional to $x^{3/2} \log x$. This arises because of the discontinuous derivative of the shear modulus at $x = 0$; a similar phenomenon is analysed in [3, §7.1].

Delale and Erdogan [19] have reduced the problem of a pressurized crack in an inhomogeneous solid, in which the shear modulus is $\mu e^{\gamma x}$ for all x, to a singular integral equation. They found the usual square-root behaviour near the crack tips. It should be possible to show further that the complete expansion contains only algebraic terms. Note that [19] also includes references to related work on cracks in other types of inhomogeneous materials.

Finally, we note that some related problems can be reduced to systems of singular integral equations. One example is the problem of §5 in which the crack is not perpendicular to the interface [20]. Another is the problem of §5 in which the two half-spaces are both anisotropic [21]. It would be of interest to extend the present method to such problems.

Appendix A

The solution cited in §3.2, due to Erdogan et al. [2], is defined by the following complicated, but elementary, functions of k.

$$
\begin{aligned}
h_1 &= kP_1 - 2\nu P_4 - 2(1-\nu), \; h_2 = kP_4, \; h_3 = kQ_1 - 2\nu Q_4 + k, \; h_4 = kQ_4, \\
P_1 &= \{(\kappa+1)^2(AD+BC) + (\kappa-1)(\kappa+3)kD - (\kappa-1)^2 B\}/Z, \\
Q_1 &= -2k\{(\kappa+1)(AD+BC) + (\kappa-1)kD + (\kappa-1)B\}/Z, \\
P_4 &= \{D(A^2+B^2-2kA-3k^2) + kB(\kappa-2-2\kappa C + (\kappa+2)(C^2+D^2))\}/Z, \\
Q_4 &= -2k\{D(A^2+B^2-k^2) + kB(-1+C^2+D^2)\}/Z, \\
Z &= \kappa D(A^2+B^2+2kA+k^2) - kB(1-2C+C^2+D^2), \\
C &= \{\delta(A+\gamma)(A^2+B^2) - (1+\delta)k^2(\delta A+\gamma) + \gamma A(A+\gamma) - \gamma B^2\}/Y, \\
D &= \{\delta B(A^2+B^2) + (1+\delta)k^2\delta B + 2\gamma AB + \gamma^2 B\}/Y, \\
Y &= k\{\delta^2(A^2+B^2) + 2\delta\gamma A + \gamma^2\}, \; R^4 = (\gamma^2+4k^2)^2 + 16\nu\gamma^2k^2/(1-\nu), \\
A &= \tfrac{1}{2}\left\{-\gamma + \sqrt{\tfrac{1}{2}(R^2+\gamma^2+4k^2)}\right\} \quad \text{and} \quad B = \tfrac{1}{2}\sqrt{\tfrac{1}{2}(R^2-\gamma^2-4k^2)}.
\end{aligned}
$$

In these expressions, $\kappa = 3-4\nu$, $\delta = 1/(1-2\nu)$ and γ is the parameter (called β in [2]) occurring in (3.4).

We require the behaviour of $g_j(k)$ for small and large values of k (we used the computer algebra system 'MAPLE'). As $k \to 0$, we find that, with an error of $O(k^4)$,

$$
\begin{aligned}
g_1 &= 5k - (16/\gamma)k^2 + (16/\gamma^2)k^3, \\
g_2 &= -3k^2 + (8/\gamma)k^3, \\
g_3 &= g_2, \\
g_4 &= 2k^3.
\end{aligned}
$$

As $k \to \infty$, we find that, with an error of $O(k^{-1})$,

$$
\begin{aligned}
g_1 &= (2-\nu)\lambda, \\
g_2 &= -\lambda k - (1-2\nu)\lambda^2,
\end{aligned}
$$

$$g_3 = g_2,$$
$$g_4 = \lambda k^2 + (1-2\nu)\lambda^2 k - \nu(3-4\nu)\lambda^3,$$

where $\lambda = \gamma/(4(1-\nu))$; the latter results agree with those in [2]. Note that $g_2 \not\equiv g_3$!

Appendix B

In order to show that $D(z)$, defined by (5.6), has just one zero in the strip $-1 < \sigma < 0$, consider the rectangular contour C, with vertices at $\pm \mathrm{i}R$ and $-1 \pm \mathrm{i}R$ ($R > 0$). Clearly, $D(z)$ is analytic within C, and so the 'principle of the argument' gives that $D(z)$ has N zeros within C, where

$$2\pi N = \text{increase in } \arg D(z) \text{ as } z \text{ goes once around } C.$$

Elementary consideration of $D(z)$, for large R, shows that $\operatorname{Re} D$ changes sign twice (near $\sigma = -\frac{1}{2}$) whereas $\operatorname{Im} D$ changes sign once (at $z = -1$); hence $N = 1$. As there is only one zero, it must be real. In fact, the existence of one real root is easily shown: note that, for all allowable materials ($m > 0$, $-1 < \nu, \nu_{\mathrm{u}} \leq \frac{1}{2}$), $D(0) > 0$, $D(-1) < 0$, $D'(0) = 0$ and $D'(\sigma)$ changes sign once in the interval $-1 < \sigma < 0$.

References

1. L.B. Freund, *Dynamic Fracture Mechanics*. Cambridge: University Press (1990).
2. F. Erdogan, A.C. Kaya and P.F. Joseph, The crack problem in bonded nonhomogeneous materials. *J. Appl. Mech.* 58 (1991) 410–418.
3. P.A. Martin, End-point behaviour of solutions to hypersingular integral equations. *Proc. Roy. Soc. A* 432 (1991) 301–320.
4. D.B. Bogy, On the plane elastostatic problem of a loaded crack terminating at a material interface. *J. Appl. Mech.* 38 (1971) 911–918.
5. A.R. Zak and M.L. Williams, Crack point stress singularities at a bi-material interface. *J. Appl. Mech.* 30 (1963) 142–143.
6. N.I. Ioakimidis, Application of finite-part integrals to the singular integral equations of crack problems in plane and three-dimensional elasticity. *Acta Mech.* 45 (1982) 31–47.
7. H.F. Bueckner, Field singularities and related integral representations. In G.C. Sih (ed.), *Mechanics of Fracture 1, Methods of Analysis and Solutions of Crack Problems* pp. 239–314. Leyden: Noordhoff (1973).
8. P.A. Martin, Exact solution of a simple hypersingular integral equation. *J. Integral Equations and Applics.* to appear.
9. G.I. Barenblatt, The mathematical theory of equilibrium cracks in brittle fracture. *Adv. in Appl. Mech.* 7 (1962) 55–129.
10. N.I. Muskhelishvili, *Singular Integral Equations*. Groningen: Noordhoff (1953).
11. A.A. Khrapkov, First fundamental problem for a piecewise-homogeneous plane with a slit perpendicular to the line of separation. *Appl. Math. & Mech. (PMM)* 32 (1968) 666–678.
12. J.G. Kuang and T. Mura, Dislocation pile-up in two-phase materials. *J. Appl. Phys.* 39 (1968) 109–120.
13. C. Atkinson, On dislocation densities and stress singularities associated with cracks and pile ups in inhomogeneous media. *Int. J. Eng. Sci.* 10 (1972) 45–71.
14. T.S. Cook and F. Erdogan, Stresses in bonded materials with a crack perpendicular to the interface. *Int. J. Eng. Sci.* 10 (1972) 677–697.
15. A.C. Kaya and F. Erdogan, On the solution of integral equations with strongly singular kernels. *Q. Appl. Math.* 45 (1987) 105–122.
16. F. Oberhettinger, *Tables of Mellin Transforms*. Berlin: Springer (1974).
17. N. Bleistein and R.A. Handelsman, *Asymptotic Expansions of Integrals*. New York: Holt, Rinehart & Winston (1975). Also, New York: Dover (1986).
18. C. Atkinson, On the stress intensity factors associated with cracks interacting with an interface between two elastic media. *Int. J. Eng. Sci.* 13 (1975) 489–504.
19. F. Delale and F. Erdogan, The crack problem for a nonhomogeneous plane. *J. Appl. Mech.* 50 (1983) 609–614.

20. N. Ashbaugh, Stress solution for a crack at an arbitrary angle to an interface. *Int. J. Fracture* 11 (1975) 205–219.
21. C. Atkinson and D.A. Eftaxiopoulos, Interaction between a crack and a free or welded boundary in media with arbitrary anisotropy. *Int. J. Fracture* 50 (1991) 159–182.